8.4 - Matrix Exponential

Recall that $e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$ This is the power series representation for the exponential function. In the same way, we can define a matrix exponential.

$$e^{3x} = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + \cdots$$

Definition: For any $n \times n$ matrix $\mathbf{A}$,

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^k\frac{t^k}{k!} + \cdots = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}$$

Example: Compute $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = AA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$A^3 = AA^2 = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad A^4 = II = I$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} + \begin{pmatrix} t^2/2 & 0 \\ 0 & t^2/2 \end{pmatrix} + \begin{pmatrix} 0 & t^3/6 \\ t^3/6 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \cdots & t + \frac{1}{6}t^3 + \frac{1}{5!}t^5 + \cdots \\ t + \frac{1}{6}t^3 + \frac{1}{5!}t^5 + \cdots & 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \cdots \end{pmatrix}$$

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$$e^{At} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

In general, $e^{A+B} \neq e^A e^B$. But <u>if</u> $AB = BA$, then $e^{A+B} = e^A e^B$.

~ A & B commute

Since A commutes with itself $(AA = AA)$,

$$e^{At} e^{-At} = e^{At-At} = e^{0 \leftarrow \text{matrix}} = I$$

so $(e^{At})^{-1} = e^{-At} = e^{A(-t)}$

So $e^{-At} = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}$

Consider $\mathbf{X}' = \mathbf{A}\mathbf{X}$. If $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$, then $\mathbf{X}' = \frac{d}{dt}\left(e^{\mathbf{A}t}\mathbf{C}\right)$. To find $\frac{d}{dt}\left(e^{\mathbf{A}t}\right)$, we use the definition:

$$\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + A^4\frac{t^4}{4!} + \cdots\right)$$

$$= A + A^2 t + \tfrac{1}{2}A^3t^2 + \tfrac{1}{6}A^4t^3 + \cdots$$

$$= A(I + At + \tfrac{1}{2}A^2t^2 + \tfrac{1}{6}A^3t^3 + \cdots)$$

$$= Ae^{At}$$

So $\vec{x} = e^{At}\vec{c} \Rightarrow \vec{x}' = Ae^{At}\vec{c} = A\vec{x}$

$\mathbf{X}' = \mathbf{A}e^{\mathbf{A}t}\mathbf{C}$

So for $\mathbf{X}' = \mathbf{A}\mathbf{X}$, with $\mathbf{A}$ containing constant entries, $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ is a solution. (Note: $\mathbf{C}$ is a column matrix of arbitrary coefficients.)

Example: Use the matrix exponential to find the general solution of the given system.

$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{X}$

For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e^{At} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$

so $\boxed{\vec{X} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \vec{C}}$

This works when A^k is nice – either has patterns or if $A^k = 0$ for some k.

Initial-value problems

$e^{\mathbf{A}t} = \mathbf{\Phi}$ is the fundamental matrix for the system, so variation of parameters yields, the general solution

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)ds$$

Note: $e^{-\mathbf{A}s}$ is $e^{\mathbf{A}t}$ with t replaced by $-s$. In our work, we will take $t_0 = 0$.

Example: Find the general solution of the given system.

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1^2 & 0 \\ 0 & 2^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1^3 & 0 \\ 0 & 2^3 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & 2t \end{pmatrix} + \begin{pmatrix} \frac{1}{2}t^2 & 0 \\ 0 & \frac{2^2}{2}t^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3!}t^3 & 0 \\ 0 & \frac{2^3}{3!}t^3 \end{pmatrix} + \cdots$$

$$e^{At} = \begin{pmatrix} 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \cdots & 0 \\ 0 & 1 + (2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$\vec{X} = e^{At}\vec{C} \Rightarrow \vec{X} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\vec{C}$$

$$\int_{t_0}^{t} e^{-As}\vec{F}(s)\,ds = \int_0^t \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} 3e^{-s} \\ -e^{-2s} \end{pmatrix} ds = \begin{pmatrix} -3e^{-s} \\ \frac{1}{2}e^{-2s} \end{pmatrix}\Bigg|_0^t$$

$$= \begin{pmatrix} -3e^{-t} + 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{pmatrix}$$

$$\vec{X} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\vec{C} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -3e^{-t} + 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \vec{X} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \vec{C} + \begin{pmatrix} -3+3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{pmatrix}$$

$$\vec{X} = \begin{pmatrix} c_1 e^t - 3 + 3e^t \\ c_2 e^{2t} + \frac{1}{2} - \frac{1}{2}e^{2t} \end{pmatrix}$$

Relabeling c_1, c_2

$$\boxed{\vec{X} = c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}}$$

Sometimes if A isn't "nice" we can make it nice.

Example: Solve the given system by diagonalizing the coefficient matrix.

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix} \mathbf{X}$$

- 1^{st} – find eigenvalues & eigenvectors

$$\begin{vmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 8\lambda + 15 = 0$$

$$\lambda = 3, 5$$

$\lambda_1 = 3$: $\begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \rightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 5$: $\begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix} \rightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

- Form a matrix P that has columns $\vec{K}_1$ & $\vec{K}_2$.

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

Find $P^{-1}AP$. $\quad P^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

$$P^{-1}AP = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 3 & 15 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

So $P^{-1}AP = D$, a diagonal matrix.

Note: then $PP^{-1}APP^{-1} = PDP^{-1} \Rightarrow A = PDP^{-1}$

and $A^k = (PDP^{-1})^k = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}}) \cdots (\cancel{P}D\cancel{P^{-1}})(\cancel{P}DP^{-1})$

$$A^k = PD^kP^{-1} \text{ with } D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \ \lambda_n^k \end{pmatrix}$$

Then $e^{At} = \overbrace{PP^{-1}}^{I} + PDP^{-1}t + \frac{1}{2}PD^2P^{-1}t^2 + \cdots$

$$e^{At} = P\,e^{Dt}\,P^{-1}$$

For $A = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix}$, $e^{At} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

So $\vec{X} = \begin{pmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{pmatrix}\vec{C}$